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On stability for impulsive perturbed systems via cone-valued Lyapunov function method

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Abstract

The notion of ϕ_0-stability recently was introduced. In this paper, we will extend this notion to the so-called eventual ϕ_0-stability for impulsive systems of differential equations under more relax conditions. Our technique depends on Lyapunov's direct method.

Keywords: Eventual ϕ_0-stability; Uniform eventual ϕ_0-stability; Eventual asymptotic ϕ_0-stability; Impulsive systems

1. Introduction

Systems of differential equations are adequate mathematical models for numerous processes and phenomena studied in biology, physics technology, etc. In recent year the mathematical theory of these systems has been developed by a great numbers of mathematicians Bainov and Simeonov [2], Lakshmikantham et al. [6], and Somoilenko and Perestyuk [10]. For more detailed bibliographies on the subject, see [3].

The main purpose of this paper is to discussed the notion of eventual ϕ_0-stability for impulsive systems of differential equations. The motivation of this work is the recent work of [4,7–9]. The paper is organized as follows.

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In Section 1, we introduce some preliminaries definitions and results which will be used throughout the paper. In Section 2, we discussed the notion of eventual ϕ_0-stability for impulsive system of differential equations.

Let $\Re^s_H$ be the s-dimensional Euclidean space with a suitable norm $\|\cdot\|$. Let $\Re^+ = [0,\infty)$, $\Re^s_H = \{x \in \Re^s : \|x\| < H\}$.

Consider the system of differential equations with impulses

$$\left.\begin{aligned} x' &= f(t,x) + R(t,y), \quad t \neq \tau_i(x,y), \quad \Delta x|_{t=\tau_i(x,y)} = A_t(x) + B_t(y),\\ y' &= h(t,x,y), \quad t \neq \tau_i(x,y), \quad \Delta y|_{t=\tau_i(x,y)} = C_t(x,y), \end{aligned}\right\} \tag{1.1}$$

where $x \in \Re^n$, $y \in \Re^m$, $f : \Re^+ \times \Re^n_H \to \Re^n$, $R : \Re^+ \times \Re^m_H \to \Re^m$, $h : \Re^+ \times \Re^n_H \times \Re^m_H \to \Re^m$, $A_t : \Re^n_H \to \Re^n$, $B_t : \Re^m_H \to \Re^m$, $C_t : \Re^n_H \times \Re^m_H \to \Re^m$, $\tau_i : \Re^n_H \times \Re^m_H \to \Re^1$.

Let $t_0 \in \Re^+$, $x_0 \in \Re^n_H$, $y_0 \in \Re^m_H$. Let $x(t,t_0,x_0,y_0)$, $y(t,t_0,x_0,y_0)$ be solutions of the system (1.1), satisfying the initial conditions $x(t_0+0,t_0,x_0,y_0) = x_0$, $y(t_0+0,t_0,x_0,y_0) = y_0$. The solution $(x(t),y(t))$ of the system (1.1) are piecewise continuous functions with points of discontinuity of the first type in which they are left continuous, i.e. at the moment t_i when the integral curve of the solution $(x(t),y(t))$ meets the hypersurface

$$\sigma_t = \{(t,x,y) \in \Re^+ \times \Re^n_H \times \Re^m_H : \ t = \tau_i(x,y)\}.$$

The following relations are satisfied:

$$x(t_i - 0) = x(t_i), \quad \Delta x|_{t=t_i} = A_t(x(t_i)) + B_t(y(t_i)),$$

$$y(t_i - 0) = y(t_i), \quad \Delta y|_{t=t_i} = C_t(x(t_i), y(t_i)),$$

together with system (1.1), we consider the following system with impulses:

$$x' = f(t,x), \quad t \neq \tau_i(x,0), \quad \Delta x|_{t=\tau_i(x,0)} = A_t(x). \tag{1.2}$$

Let

$$s_t = \{(t,x) \in \Re^+ \times \Re^n_H : t = \tau_i(x,0)\}.$$

We introduce the following definition depending on that given in [1,5]:

Definition 1. A proper subset K_1 of $\Re^n$ or a proper subset K_2 of $\Re^m$ is called a cone if

(i) $\lambda K_i \subset K_i$, $\lambda \geqslant 0$,
(ii) $K_i + K_i \subset K_i$,
(iii) $\overline{K}_i = K_i$,
(iv) $K_i^\circ \neq \emptyset$,
(v) $K_i \cap (-K_i) = \{0\}$, $i = 1,2$.

where $\overline{K}_i$ and K_i° denote the closure and interior of K_i respectively, and ∂K_i denotes the boundary of K_i, $i = 1, 2$, it follows that $K = K_1 \cup K_2 \subset \Re^n \cup \Re^m$ be a cone in $\Re^n \cup \Re^m$.

We introduce the following definitions as given in [1,5]:

Definition 2. The set K^* is called the adjoint cone if

$$K^* = \{\phi \in \Re^n \cup \Re^m : (\phi, x + y) \geqslant 0 \quad \text{for } x \in K_1 \subset K, y \in K_2 \subset K\}$$

satisfies the properties (i)–(v) of Definition 1, where $(\phi, x + y) \leqslant \|\phi\|(\|x\| + \|y\|)$. For $m > n$, and $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_m)$. Thus

$$\begin{aligned} x + y &= (x_1, x_2, \dots, x_n,) + (y_1, y_2, \dots, y_m) \\ &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n, y_{n+1}, \dots, y_m), \end{aligned}$$

$x \in \partial K_i$ iff $(\phi, x) = 0$ for some $\phi \in K_{i0}^*$, $K_{i0}^* = K_{i0} - \{0\}$, $i = 1, 2$.

Definition 3. A function $F : D \to \Re^n$, $D \subset \Re^n$ is called quasimonotone relative to the cone K_i, $i = 1, 2$, if $x, y \in D$ and $y - x \in \partial K_i$, then there exists $\phi_0 \in K_{i0}^*$ such that $(\phi_0, y - x) = 0$ and $(\phi_0, F(y) - F(x)) \geqslant 0$.

Definition 4. A function $\psi(r)$ is said to be belong the class $\mathscr{K}$ if $\psi \in C[\Re^+, \Re^+]$, $\psi(0) = 0$, and $\psi(r)$ is strictly monotone increasing in r.

Let $\tau_0(x, y) = 0$ for $(x, y) \in \Re_H^n \times \Re_H^m$. Following [4] we define the sets

$$\Gamma_i = \{(t, x, y) \in \Re^+ \times \Re_H^n \times \Re_H^m : \tau_{i-1}(x, y) < t < \tau_i(x, y)\},$$

$$\Omega_i = \{(t, x) \in \Re^+ \times \Re_H^n : \tau_{i-1}(x, 0) < t < \tau_i(x, 0)\}.$$

As in [4], we use the classes $\mathscr{V}_0$ and $\mathscr{W}_0$ of piecewise continuous functions which are analogue to Lyapunov functions.

Definition 5 [4]. We say that the function $V : \Re^+ \times \Re_H^n \times \Re_H^m \to K$ belongs to the class $\mathscr{V}_0$ if the following conditions hold:

(1) The function V is continuous in $\bigcup_{i=1}^\infty \Gamma_i$ and is locally Lipschitzian with respect to x and y in each of the sets Γ_i.
(2) $V(t, 0, 0) = 0$ for $t \in \Re^+$.
(3) For each $i = 1, 2, \dots$ and for any point $(t_0, x_0, y_0) \in \sigma_t$ there exist the finite limits

$$V(t_0 - 0, x_0, y_0) = \lim_{\substack{(t,x,y) \to (t_0, x_0, y_0) \\ (t,x,y) \in G_i}} V(t, x, y),$$

$$V(t_0+0,x_0,y_0) = \lim_{\substack{(t,x,y)\to(t_0,x_0,y_0)\\(t,x,y)\in G_{i+1}}} V(t,x,y),$$

and the equality $V(t_0-0,x_0,y_0) = V(t_0,x_0,y_0)$ holds.

(4) For any point $(t,x,y)\in\sigma_t$, the following inequality holds:

$$V(t+0,x+A_t(x)+B_t(y),y+C_t(x,y)) \leqslant V(t,x,y). \tag{1.3}$$

Definition 6 [4]. We say that the function $W : I\times\Re^n_H \to K$ belongs to the class $\mathscr{W}_0$ if the following conditions hold:

(1) The function W is continuous in $\bigcup_{i=1}^{\infty}\Omega_i$ and is locally Lipschitz with respect to x in each of the sets Ω_i.
(2) $W(t,0)=0$ for $t\in\Re^+$.
(3) There exist the finite limits

$$W(t_0-0,x_0) = \lim_{\substack{(t,x)\to(t_0,x_0)\\(t,x)\in\Omega_i}} W(t,x),$$

$$W(t_0+0,x_0) = \lim_{\substack{(t,x)\to(t_0,x_0)\\(t,x)\in\Omega_i}} W(t,x),$$

and the equality $W(t_0-0,x_0) = W(t_0,x_0)$ holds.

(4) For any point $(t,x)\in s_t$, the following inequality holds:

$$W(t+0,x+A_t(x)) \leqslant W(t,x). \tag{1.4}$$

Let $V\in\mathscr{V}_0$, and $x(t)$, $y(t)$ be the maximal solution of (1.1), for $(t,x,y)\in\bigcup_{i=1}^{\infty}\Gamma_i$, following [4] we define

$$V_1'(t,x,y) = \lim_{s\to 0}[V(t+s,x+s(f(t,x)+g(t,y)),y+sh(t,x,y)) - V(t,x,y)],$$

and

$$V'_{(2.1)}(t,x,y) = D^+V(t,x,y), \quad t\neq\tau_i(x,y),$$

where $D^+V(t,x,y)$ is the upper right Dini derivative of the function $V(t,x,y)$.

Analogously one can define the function $W'_{(2.2)}(t,x)$ for an arbitrary function $W\in\mathscr{W}_0$ for $(t,x)\in\bigcup_{i=1}^{\infty}\Omega_i$. The following definition is new and related with that of [1,4]:

Definition 7. The zero solution of system (1.1) is said to be eventual ϕ_0-equi-stable if for all $\epsilon>0$, for all $t_0\in\Re^+$, there exists $\delta=\delta(t_0,\epsilon)>0$, for all $(x_0,y_0)\in(\Re^n_H\times\Re^m_H)$, such that

$$(\phi_0, x_0 + y_0) < \delta$$
$$\text{implies } (\phi_0, x(t, t_0, x_0, y_0) + y(t, t_0, x_0, y_0)) < \epsilon, \quad t \geqslant t_0 \geqslant \tau_0,$$

where $\phi_0 \in K_0^*$. In the case of uniformly eventually ϕ_0-stable, δ independent of t_0.

Any eventual ϕ_0-equistability concepts can be similarly defined.

Definition 8. We say conditions (A) hold if the following conditions are satisfied:

(A_1) The functions $f(t,x)$, $R(t,y)$ and $h(t,x,y)$ are continuous in their definitions domains and $f(t,x)$ is quasimonotone in x relative to the cone K_1, $R(t,y)$ is quasimonotone in y relative to the cone K_2 and $h(t,x,y)$ is quasimonotone in x relative to the cone K_1, quasimonotone in y relative to the cone K_2, $f(t,0) = R(t,0) = 0$, $h(t,0,0) = 0$ for $t \in \mathfrak{R}^+$.

(A_2) There exists a constant $L > 0$ such that

$$h(t,x,y) \leqslant L, \quad (t,x,y) \in \mathfrak{R}^+ \times \mathfrak{R}_H^n \times \mathfrak{R}_H^m.$$

(A_3) There exists a continuous function $P : I \to I$ such that $P(0) = 0$ and $\|R(t,y)\| \leqslant P(\|y\|)$ for $(t,x) \in \mathfrak{R}^+ \times \mathfrak{R}_H^n$.

(A_4) The functions A_t, B_t, C_t are continuous in their definitions domains and $A_t(0) = B_t(0) = C_t(0,0) = 0$.

(A_5) If $x \in \mathfrak{R}_H^n$ and $y \in \mathfrak{R}_H^n$, then $\|x + A_t(x) + B_t(y)\| \leqslant \|x\|$, and $\|y + c_t(x,y)\| \leqslant \|y\|$, $i = 1, 2$.

(A_6) The functions $\tau_i(x,y)$ are continuous and for $(x,y) \in \mathfrak{R}_H^n \times \mathfrak{R}_H^m$ the following relations hold: $0 < \tau_1(x,y) < \tau_2(x,y) < \cdots < \lim_{t\to\infty} \tau_i(x,y) = \infty$ uniformly in $\mathfrak{R}_H^n \times \mathfrak{R}_H^m$, and

$$\inf_{\mathfrak{R}_H^n \times \mathfrak{R}_H^m} \tau_{i+1}(x,y) - \sup_{\mathfrak{R}_H^n \times \mathfrak{R}_H^m} \tau_i(x,y) \geqslant \theta > 0, \quad i = 1, 2, \ldots.$$

(A_7) For each point $(t_0, x_0, y_0) \in \mathfrak{R}^+ \times \mathfrak{R}_H^n \times \mathfrak{R}_H^m$ the solution $x(t, t_0, x_0)$, $y(t, t_0, x_0, y_0)$ of the system (2.1) is unique and defined in (t_0, ∞).

(A_8) For each point $(t_0, x_0) \in \mathfrak{R}^+ \times \mathfrak{R}_H^n$ the solution $x(t, t_0, x_0)$ of system (1.2) satisfying $x(t_0 + 0, t_0, x_0) = x_0$ is unique and exists for all $t \in (t_0, \infty)$.

(A_9) The integral curve of each solution of system (1.1) meets each of the hypersurfaces $\{\sigma_t\}$ at most once.

2. Main results

In this section, we give a partial generalization of the work of Kulev and Bainov [4].

Theorem 1. *Assume that*

(H_1) *The condition* (A) *holds.*
(H_2) *There exist functions* $V \in \mathscr{V}_0$, $a, b \in \mathscr{K}$ *such that*

$$a(\phi_0, x+y) \leqslant (\phi_0, V(t,x,y)) \leqslant b(\phi_0, x+y), \quad (t,x,y) \in \Re^+ \times \Re^n_H \times \Re^m_H.$$

(H_3) $V'_{(2.1)}(t,x,y) \leqslant G(t, V(t,x,y))$ *for* $(t,x,y) \in \bigcup_{i=1}^{\infty} \Gamma_i$,

$$\int_{t_k}^{t_{k+1}} G(s, V(s,x))\,\mathrm{d}s \leqslant V(t_0, x_0, y_0), \quad [t_k, t_{k+1}] \subset (t_0, \infty),$$

where $G \in C[\Re^+, \Re^+]$ *and* $G(t,0) = 0$.

Then the zero solution of the system (2.1) *is uniformly eventually* ϕ_0*-stable.*

Proof. Let $0 < \epsilon < H$ and $t_0 \in \Re^+$, assume that $t_0 \leqslant \tau_1(x,y)$ for $(t,x) \in \Re^n_H \times \Re^m_H$. Since $V(t,0,0) = 0$ and from Definition 5, it follows that there exists a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that

$$\|x_0\| + \|y_0\| < \delta_1 \ \text{implies}\ V(t_0+0, x_0, y_0) < a_1(\epsilon), \quad a_1 \in \mathscr{K}.$$

Now for some $\phi_0 \in K_0^*$,

$$\|\phi_0\|\|x_0 + y_0\| < \|\phi_0\|\delta_1 \ \text{implies}\ \|\phi_0\|\|V(t_0+0, x_0, y_0)\| < \|\phi_0\| a_1(\epsilon).$$

Thus

$$(\phi_0, x_0 + y_0) \leqslant \|\phi_0\|\|x_0 + y_0\| < \|\phi_0\|\delta_1$$

implies

$$(\phi_0, V(t_0+0, x_0, y_0)) \leqslant \|\phi_0\|\|V(t_0+0, x_0, y_0)\| < \|\phi_0\| a_1(\epsilon).$$

Thus it follows that

$$(\phi_0, x_0 + y_0) < \delta \ \text{implies}\ (\phi_0, V(t_0+0, x_0, y_0)) < a(\epsilon), \tag{2.1}$$

where $\|\phi_0\|\delta_1 = \delta$ and $\|\phi_0\| a_1(\epsilon) = a(\epsilon)$.

Let $x_0 \in \Re^n_H$, $y_0 \in \Re^m_H$, $(\phi_0, x_0 + y_0) < \delta$ and let $x(t) = x(t, t_0, x_0, y_0)$, $y(t) = y(t, t_0, x_0, y_0)$ be the maximal solution of (1.1). Then by (H_2) and for $\epsilon > 0$, let $\delta = b^{-1}[a(\epsilon)/2]$ independent of t_0 for $a, b \in \mathscr{K}$. Let $x_0 \in \Re^n_H$, $y_0 \in \Re^m_H$, $(\phi_0, x_0 + y_0) < \delta$ and from (1.3) and (H_3), we get

$$
\begin{aligned}
a(\phi_0, x(t)+y(t)) &\leqslant (\phi_0, V(t,x,y)) \\
&\leqslant (\phi_0, V(t_0+0, x_0, y_0)) + \left(\phi_0, \int_{t_k}^{t_{k+1}} G(s, V(s,x))\,\mathrm{d}s\right) \\
&\leqslant b(\phi_0, V(t_0+0, x_0, y_0)) + (\phi_0, V(t_0, x_0, y_0)) \\
&\leqslant\ < b(\phi_0, x_0+y_0) \\
&\leqslant b(\delta) \\
&\leqslant a(\epsilon)
\end{aligned}
$$

for $t \geqslant t_0 \geqslant \tau_0$. Therefore $(\phi_0, x(t)+y(t)) < \epsilon$. Hence the zero solution of system (1.1) is uniformly eventually ϕ_0-stable. □

Theorem 2. *Suppose that the assumptions of Theorem* 1 *be satisfied except the condition* (H_3) *being replaced by the condition*

(H_4) $(\phi_0, V_{(2.1)}(t,x,y))' \leqslant -g(t)(\phi_0, y), \quad (t,x,y) \in \mathfrak{R}^+ \times \mathfrak{R}^n_H \times \mathfrak{R}^m_H,\ g \in \mathscr{K}.$

We further assume that

(H_5) *There exist functions $W \in \mathscr{W}_0$ and $a_1, b_1 \in \mathscr{K}$ such that*

$$a_1(\phi_0, x) \leqslant W(t,x) \leqslant b_1(\phi_0, x), \quad (t,x) \in \mathfrak{R}^+ \times \mathfrak{R}^n_H.$$

(H_6) *There exist functions $W \in \mathscr{W}_0$ and $g_1(t) \in \mathscr{K}$ such that*

$$W'_{(2.2)}(t,x) \leqslant -g_1(t)(W(t,x)), \quad (t,x) \in \bigcup_{i=1}^{\infty} \Omega_i.$$

(H_7) $\|W(t,x_1) - W(t,x_2)\| \leqslant \ell \|x_1 - x_2\|.$

If the zero solution of (2.1) *is equi-asymptotically ϕ_0-stable, then it is uniformly eventually asymptotically ϕ_0-stable.*

Proof. From the assumption of equi-asymptotically ϕ_0-stable, then for $\epsilon > 0$, $\tau > 0$, there exist $\delta > 0$, $T > 0$ such that

$$(\phi_0, x_0+y_0) < \delta \text{ implies } (\phi_0, x(t)+y(t)) < \epsilon, \quad t \geqslant t_0 + T, t_0 \geqslant \tau_0, \tag{2.2}$$

where $T = (n-1)[T_1(\epsilon) + T_2(\epsilon)]$.

By Theorem 1, it follows that the zero solution of the system (1.1) is uniformly eventually ϕ_0-stable. Therefore for any $0 < \epsilon \leqslant M$, there exists $\delta = \delta(\epsilon)$ such that

$$(\phi_0, x+y) < \epsilon \quad \text{whenever } (\phi_0, x_0+y_0) < \delta,\ t > t_0 \geqslant \tau_0. \tag{2.3}$$

Going through as in [4] Theorem 1 we choose $\delta_1 = \delta_1(\epsilon) > 0$ such that $\delta_1(\epsilon) < \frac{1}{2}\delta(\epsilon)$, and

$$P(s) < \frac{1}{2d} g_1\left(a_1\left(\frac{1}{2}\delta\right)\right) \quad \text{for } 0 \leqslant s \leqslant \delta_1, \tag{2.4}$$

where ℓ is Lipschitz constant for the function W. Moreover let $T_1 = T_1(\epsilon) > 0$ and $T_2 = T_2(\epsilon) > 0$ such that

$$T_1(\epsilon) > \frac{b(M) - a\left(\frac{1}{2}\delta_1\right)}{g\left(\frac{1}{2}\delta_1\right)}, \tag{2.5}$$

$$T_2(\epsilon) > \frac{2\left[b_1(M) - a_1\left(\frac{1}{2}\delta_1\right)\right]}{g_1\left(a_1 \frac{1}{2}\delta_1\right)}. \tag{2.6}$$

Let the positive integer v be such that

$$b(H) - (v-1)\frac{\delta_1 g\left(\frac{1}{2}\delta_1\right)}{2L} < 0. \tag{2.7}$$

Let $t_0 \in \Re^+$, $(\phi_0, x_0 + y_0) < \delta$. Assume that for all $t \in [t_0, t_0 + T_1]$ the inequality $(\phi_0, y(t)) \geqslant \frac{1}{2}\delta_1$ holds. Then from (H_4), we obtain

$$(\phi_0, V_{(2.1)}(t,x,y))' \leqslant -g(\phi_0, y(t)) \leqslant -g\left(\frac{1}{2}\delta_1\right),$$
$$t \in (t_0, t_0 + T_1], \quad t \neq \tau_i(x(t), y(t)), \quad i = 1, 2, \ldots \tag{2.8}$$

By integrating (2.8) on $[t_0, t_0 + T_1]$ and making use of (1.3), (H_3), and (2.5) we get

$$\begin{aligned}
a\left(\frac{1}{2}\delta_1\right) &\leqslant a(\phi_0, x(t_0 + T_1) + y(t_0 + T_1)) \\
&\leqslant (\phi_0, V(t_0 + T_1, x(t_0 + T_1), y(t_0 + T_1))) \quad \text{from } (\mathrm{H}_3) \\
&\leqslant (\phi_0, V(t_0 + 0, x_0, y_0)) - g\left(\frac{1}{2}\delta_1\right)T_1 \quad \text{from integrating (2.8)} \\
&\leqslant b(\phi_0, x_0 + y_0) - g\left(\frac{1}{2}\delta_1\right)T_1 \quad \text{from } (\mathrm{H}_3) \text{ and (1.3)} \\
&\leqslant b(\delta) - g\left(\frac{1}{2}\delta_1\right)T_1 \\
&\leqslant b(M) - g\left(\frac{1}{2}\delta_1\right)\frac{b(M) - a\left(\frac{1}{2}\delta_1\right)}{g\left(\frac{1}{2}\delta_1\right)} \quad \text{from (2.5)} \\
&= a\left(\frac{1}{2}\delta_1\right),
\end{aligned}$$

which is a contradiction. Thus there exists ξ_1, $t_0 < \xi_1 < t_0 + T_1$ such that

$$(\phi_0, y(\xi_1)) < \frac{1}{2}\delta_1, \quad \text{i.e., } \|y(\xi_1)\| \leqslant \frac{\frac{1}{2}\delta_1}{\|\phi_0\|} < \frac{1}{2}\delta_1. \tag{2.9}$$

To prove that for any $t \in [\xi_1, t_0 + T_1 + T_2]$ the inequality $(\phi_0, y(t)) < \delta_1 < \frac{1}{2}\delta$ holds, then there exists $\xi_2 \in [\xi_1, t_0 + T_1 + T_2]$ such that $(\phi_0, x(\xi_2)) < \frac{1}{2}\delta$. Suppose this is false, then by (H_5), (H_6), and A_3, in view of (2.4) we obtain

$$W'_{(2.1)}(t, x) \leqslant -\frac{1}{2} g_1\left(a_1\left(\frac{1}{2}\delta\right)\right) \quad \text{for } t \in [\xi_1, t_0 + T_1 + T_2], \; t \neq \tau_i(x(t), y(t)). \tag{2.10}$$

By integrating (2.10) on $[\xi_1, t_0 + T_1 + T_2]$ and making use of (H_5), (1.4), and (2.6), thus there exists $\xi_2 \in [\xi_1, t_0 + T_1 + T_2]$ such that

$$(\phi_0, x(\xi_2)) < \frac{1}{2} \quad \text{then } (\phi_0, x(\xi_2) + y(\xi_2)) < \frac{1}{2}\delta + \frac{1}{2}\delta < \delta.$$

Now, it follows from uniform ϕ_0-stability that if

$$(\phi_0, x(t) + y(t)) < \epsilon \quad \text{for } t > \xi_2$$

holds, then

$$(\phi_0, x(t) + y(t)) < \epsilon \quad \text{for } t > t_0 + T_1(\epsilon) + T_2(\epsilon).$$

Now, let us suppose that there exists $\xi_3 \in [\xi_1, t_0 + T_1 + T_2]$ for which $(\phi_0, y(\xi_3)) \geqslant \delta_1$ and let $\xi_5 = \inf\{t \in [\xi_1, t_0 + T_1 + T_2] : (\phi_0, y(t)) \geqslant \delta_1\}$, then $(\phi_0, y(\xi_5)) \leqslant \delta_1$ and $(\phi_0, y(t)) < \delta$ for $t \in [\xi_1, \xi_5]$. If $(\phi_0, y(\xi_5)) < \delta_1$, then from the definition of ξ_5, it follows that $(\phi_0, y(\xi_5 + 0)) \geqslant \delta_1$, hence $\xi_5 = \tau_r(x(\xi_5), y(\xi_5))$ for some positive integer r. But then from (A_5), we obtain that

$$(\phi_0, y(\xi_5 + 0)) = (\phi_0, y(\xi_5) + C_r(x(\xi_5), y(\xi_5))) \leqslant \|\phi_0\| \|y(\xi_5)\| < \delta_1,$$

which is a contradiction. Thus

$$(\phi_0, y(\xi_5)) = \delta_1, \quad \xi_5 \neq \tau_i(x(\xi_5), y(\xi_5)), \; i = 1, 2, \ldots$$

Now using (A_5), we conclude that there exists ξ_4, $\xi_1 < \xi_4 < \xi_5 < t_0 < T_1 + T_2$, such that $\xi_4 \neq \tau_i(x(\xi_4), y(\xi_4))$, $i = 1, 2, \ldots,$

$$(\phi_0, y(\xi_4)) = \frac{1}{2}\delta_1 \quad \text{and} \quad \frac{1}{2}\delta_1 < (\phi_0, y(t)) < \delta_1 \;\; \text{for } t \in (\xi_4, \xi_5),$$

by (A_2), it follows that

$$\frac{d}{dt}\|y(t)\| \leqslant L \quad \text{for } t \neq \tau_i(x(t), y(t)), \; i = 1, 2, \ldots$$

Since the zero solution of (1.1) is equi-asymptotically ϕ_0-stable, we can obtain that $\xi_5 - \xi_4 \geqslant \frac{\delta_1}{2L}$. Thus by ($H_4$), it follows that

$$(\phi_0, V(t,x,y))' \leqslant -g(\phi_0,y) \leqslant -g\left(\frac{1}{2}\delta_1\right) \quad \text{for } t \in [\xi_4, \xi_5],$$
$$t \neq \tau_i(x(t),y(t)),\ i = 1,2,\ldots \tag{2.11}$$

Also by integrating (2.11) and making use of (1.3), we obtain

$$\begin{aligned}(\phi_0, V(\xi_5,x(\xi_5),y(\xi_5))) &\leqslant (\phi_0, V(\xi_4,x(\xi_4),y(\xi_4))) - g\left(\frac{1}{2}\delta_1\right)(\xi_4 - \xi_5)\\ &\leqslant (\phi_0, V(\xi_4,x(\xi_4),y(\xi_4))) - g\left(\frac{1}{2}\delta_1\right)\frac{\delta_1}{2L}.\end{aligned}$$

Thus we have proved that if $(\phi_0, x_0 + y_0) < \delta$, the following two cases are possible:

(1) $(\phi_0, x(t) + y(t)) < \epsilon$, $t \geqslant t_0 + T_1 + T_2$, or
(2) there exist ξ_4, ξ_5, $t_0 < \xi_4 < \xi_5 < t_0 + T_1 + T_2$ such that

$$(\phi_0, V(\xi_5,x(\xi_5),y(\xi_5))) \leqslant (\phi_0, V(\xi_4,x(\xi_4),y(\xi_4))) - g\left(\frac{1}{2}\delta_1\right)\frac{\delta_1}{2L}.$$

In the same way we prove that one of the following two possibilities takes place:

(1) $(\phi_0, x(t) + y(t)) < \epsilon$, $t \geqslant t_0 + 2[T_1(\epsilon) + T_2(\epsilon)]$, or
(2) there exist ξ_9, ξ_{10}, $t_0 + T_1 + T_2 < \xi_9 < \xi_{10} < t_0 + 2[T_1 + T_2]$ such that

$$(\phi_0, V(\xi_{10},x(\xi_{10}),y(\xi_{10}))) \leqslant (\phi_0, V(\xi_9,x(\xi_9),y(\xi_9))) - g\left(\frac{1}{2}\delta_1\right)\frac{\delta_1}{2L}.$$

By induction we can prove that if $(\phi_0, x(t) + y(t)) < \delta$, we have one of the following two cases:

(1) $(\phi_0, x(t) + y(t)) < \epsilon$, $t \geqslant t_0 + (n-1)[T_1(\epsilon) + T_2(\epsilon)]$, or
(2) there exist ξ_{5n-1}, ξ_{5n}, $t_0 + (n-1)[T_1 + T_2] < \xi_{5n-1} < \xi_{5n} < t_0 + n[T_1 + T_2]$ such that

$$(\phi_0, V(\xi_{5n},x(\xi_{5n}),y(\xi_{5n}))) \leqslant (\phi_0, V(\xi_{5n-1},x(\xi_{5n-1}),y(\xi_{5n-1}))) - g\left(\frac{1}{2}\delta_1\right)\frac{\delta_1}{2L}.$$

If for any positive integer $n \geqslant v$ the second one holds, then by $\xi_{5(n-1)} < t_0 + (n-1)[T_1 + T_2] < \xi_{5n-1}$. Thus from ($H_4$), and (2.7), we obtain

$$
\begin{aligned}
(\phi_0, V(\xi_{5v}, x(\xi_{5v}), y(\xi_{5v}))) &\leqslant (\phi_0, V(\xi_{5v-1}, x(\xi_{5v-1}), y(\xi_{5v-1}))) - g\left(\frac{1}{2}\delta_1\right)\frac{\delta_1}{2L} \\
&\leqslant (\phi_0, V(\xi_{5(v-1)}, x(\xi_{5(v-1)}), y(\xi_{5(v-1)}))) - g\left(\frac{1}{2}\delta_1\right)\frac{\delta_1}{2L} \\
&\leqslant (\phi_0, V(\xi_{5(v-1)-1}, x(\xi_{5(v-1)-1}), y(\xi_{(5v-1)-1}))) - g\left(\frac{1}{2}\delta_1\right)\frac{2\delta_1}{2L} \\
&\leqslant \cdots \leqslant (\phi_0, V(\xi_4, x(\xi_4), y(\xi_4))) - g\left(\frac{1}{2}\delta_1\right)\frac{(v-1)\delta_1}{2L} \\
&\leqslant b(M) - g\left(\frac{1}{2}\delta\right)\frac{v-1}{2L}\delta_1 < 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
&(\phi_0, x_0 + y_0) < \delta \text{ implies } (\phi_0, x(t) + y(t)) < \epsilon, \\
&\quad t \geqslant t_0 + (n-1)[T_1(\epsilon) + T_2(\epsilon)].
\end{aligned}
$$

We can take $T = (n-1)[T_1(\epsilon) + T_2(\epsilon)]$. Thus

$$
(\phi_0, x_0 + y_0) < \delta \text{ implies } (\phi_0, x(t) + y(t)) < \epsilon, \quad t \geqslant t_0 + T.
$$

Then the zero solution of the (1.1) is uniformly eventually asymptotically ϕ_0-stability and the proof is completed. □

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